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LIST OF DEGENERATE FINITE-DIMENSIONAL QUASIREPRESENTATIONS OF GROUPS

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Abstract. We apply well-known results concerning finite-dimensional quasi-representations
of groups to make a list of finite-dimensional quasi-representations some of whose characterizing subspaces vanish.

§ 1. Introduction

Recall that any mapping T of a given group G into the group of invertible operators on some Banach space E such that $T(e_G) = 1_E$ (where $E_G = e$ stands for the identity element of G) and the norm

$$\|T(g_1g_2) - T(g_1)T(g_2)\|, \qquad g_1, g_2 \in G,$$

is uniformly small on G, which means that

$$\|T(g_1g_2) - T(g_1)T(g_2)\| \leq \delta \text{ for any } g_1, g_2 \in G \text{ and for some small } \delta > 0,$$

is referred to as a *quasi-representation* (more exactly, as a δ-*quasi-representation*); see [1–3].

Let us recall the general form of finite-dimensional representations of arbitrary groups ([1–3]).

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Theorem 1. *Let G be a group and let T be a quasi-representation of G on a finite-dimensional vector space E_T. Let E_T^* be the conjugate space of E_T. Let L be the set of vectors $\xi \in E_T$ for which the orbit $\{T(g)\,\xi \mid g \in G\}$ is bounded in E_T and let M be the set of functionals $f \in E_T^*$ for which the orbit $\{T(g)^* f \mid g \in G\}$ is bounded in E_T^*; then L and the annihilator $M^\perp$ are vector subspaces of E_T which are T-invariant. Consider the collection of subspaces $\{0\}$, $L \cap M^\perp$, $M^\perp$, $L + M^\perp$, and $E = E_T$ (in ascending order) and write the matrix $t(g)$ of the operator $T(g)$, $g \in G$, in block form (with respect to the decomposition of the space E into the direct sum of subspaces $L \cap M^\perp$, $M^\perp \setminus (L \cap M^\perp)$, $L \setminus (L \cap M^\perp)$, and $E \setminus (L + M^\perp)$), where the symbol "$\setminus$" is used to denote a complementary subspace*:

$$t(g) = \begin{pmatrix} \alpha(g) & \varphi(g) & \sigma(g) & \tau(g) \\ 0 & \beta(g) & 0 & \rho(g) \\ 0 & 0 & \gamma(g) & \chi(g) \\ 0 & 0 & 0 & \delta(g) \end{pmatrix}, \qquad g \in G. \tag{1}$$

(*Here we have $t_{23}(g) = 0$, because L is invariant under T.) Then the following assertions hold*:

1) *the mappings α, δ, γ, σ, and χ are bounded*;

2) *the mappings t_1 and t_2 defined by the equations*

$$t_1(g) = \begin{pmatrix} \alpha(g) & \varphi(g) \\ 0 & \beta(g) \end{pmatrix}, \qquad t_2(g) = \begin{pmatrix} \beta(g) & \rho(g) \\ 0 & \delta(g) \end{pmatrix},$$

are representations of the group G;

3) *the mapping τ is a quasi-cocycle with respect to the representations t_1 and t_2, i.e., the mapping $(g,h) \to \tau(gh) - \alpha(g)\tau(h) - \varphi(g)\rho(h) - \tau(g)\delta(h)$, $g, h \in G$, is bounded.*

In this note, a quasi-representation T of G in a finite-dimensional vector space is said to be *nondegenerate* if all spaces subspaces $L \cap M^\perp$, $M^\perp \setminus (L \cap M^\perp)$, $L \setminus (L \cap M^\perp)$, and $E \setminus (L + M^\perp)$ are nonzero and *degenerate* otherwise, and we discuss the structure of degenerate quasi-representations.

Here and below, we use the definitions, notions, and notation of [3] without additional comments.

§ 2. Main Theorem

Theorem. *Consider the subspaces $L \cap M^\perp$, $M^\perp \setminus (L \cap M^\perp)$, $L \setminus (L \cap M^\perp)$, and $E \setminus (L + M^\perp)$ (of E) introduced above. Then the following possibilities occur*:

(1) *$L \cap M^\perp = \{0\}$ and the other three spaces are nonzero.*

(2) $M^{\perp} \setminus (L \cap M^{\perp}) = \{0\}$ *and the other three spaces are nonzero.*
(3) $L \setminus (L \cap M^{\perp}) = \{0\}$ *and the other three spaces are nonzero.*
(4) $L + M^{\perp} = E = \{0\}$ *and the first three spaces are nonzero.*
(5) $L = \{0\}$ *and* $M = \{0\}$*.*
(6) $L = E$ *and* $M = E$*.*
(7) $L = \{0\}$ *and* $M = E$*.*
(8) $L = E$ *and* $M = \{0\}$*.*
(9) $L = \{0\}$ *and* $\{0\} \neq M \neq E$*.*
(10) $L = E$ *and* $\{0\} \neq M \neq E$*.*
(11) $\{0\} \neq L \neq E$ *and* $M = E$*.*
(12) $\{0\} \neq L \neq E$ *and* $M = \{0\}$*.*

The corresponding form of the quasi-representation T *is given in the list below.*

Proof. Let us present the list of models for the above list of situations.

(1) If $L \cap M^{\perp} = \{0\}$ and the other three spaces are nonzero, then the first row and the first column of the matrix (1) are absent, and the matrix has 3×3 block form.
(2) If $M^{\perp} \setminus (L \cap M^{\perp})$ vanishes and the other three spaces are nonzero, then the second row and the second column of the matrix (1) are absent, and the matrix has 3×3 block form.
(3) If $L \setminus (L \cap M^{\perp})$ vanishes and the other three spaces are nonzero, then the third row and the third column of the matrix (1) are absent, and the matrix has 3×3 block form. It should be noted that, in this case, the defect matrices $T(gh) - T(g)T(h)$ can be nonzero only at the submatrix in the upper right corner, which can happen indeed for a quasi-representation of a group having an unbounded character ρ and a nontrivial quasi-character τ and given by $g \mapsto \begin{pmatrix} 1 & 0 & \tau(g) \\ 0 & \beta(g) & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
(4) If $L + M^{\perp} = E$ and the first three spaces are nonzero, then the fourth row and the fourth column of the matrix (1) are absent, and the matrix has 3×3 block form.
(5) If $L = \{0\}$ and $M = \{0\}$, then T is an ordinary representation all of whose nonzero vectors have unbounded orbits.
(6) If $L = E$ and $M = E$, then T is a bounded quasi-representation of G in E (which can be an ordinary representation).
(7) If $L = \{0\}$ and $M = E$, then T is a zero 0-dimensional mapping.
(8) If $L = E$ and $M = \{0\}$, then T is a zero 0-dimensional mapping.

(9) If $L = \{0\}$ and $\{0\} \neq M \neq E$, then the first and third rows and the first and third columns of the matrix (1) are absent, and the matrix has 2×2 block form defining an ordinary representation of G.

(10) If $L = E$ and $\{0\} \neq M \neq E$, then the second and fourth rows and the second and fourth columns of the matrix (1) are absent, and the matrix has 2×2 block form.

(11) If $\{0\} \neq L \neq E$ and $M = E$, then the first and second rows and the first and second columns of the matrix (1) are absent, and the matrix has 2×2 block form.

(12) If $\{0\} \neq L \neq E$ and $M = \{0\}$, then the third and fourth rows and the third and fourth rows of the matrix (1) are absent, and the matrix has 2×2 block form defining an ordinary representation of G.

All these assertions follow immediately from the theorem cited in the introduction.

Example. For the one-dimensional quasi-representations, see items (5) and (6) of the theorem. If T is a 2×2 quasi-representation of a group G, then the admissible possibilities for T are listed in items (5)–(12) of the theorem. If T is a finite-dimensional quasi-representation of a group G of dimension at least three, then all possibilities (1)–(12) are admissible.

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